Topology and its Applications 47 (1992) 115–127
North-Holland

Monotone normality

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Received 26 March 1991
Revised 25 October 1991

Abstract

Balogh, Z. and M.E. Rudin, Monotone normality, Topology and its Applications 47 (1992) 115–127.

Two theorems are given analyzing the possible refinements of open covers of a monotonically normal space X. The first shows that X is paracompact if and only if X has no closed subset homeomorphic to a stationary subset of a regular uncountable cardinal. The second shows that if $\mathscr{U}$ is an open cover of X, then $\mathscr{U}$ has a σ-disjoint open, partial refinement $\mathscr{V}$ such that $X - \bigcup\mathscr{V}$ is the union of a discrete family of stationary subsets of regular uncountable cardinals.

Keywords: Monotonically normal, paracompact, refinement, shrinking.

AMS (MOS) Subj. Class.: Primary 54D18, 54B10, 54D15, 54A25, 54D45.

A topological space X is said to be *monotonically normal* if it has a Hausdorff topology $\mathscr{T}$ and a function $H: X \times \mathscr{T} \to \mathscr{T}$ such that $x \in U \in \mathscr{T}$ and $y \in V \in \mathscr{T}$ imply that $x \in H(x, U) \subset U$, $y \in H(y, V) \subset V$, and $H(x, U) \cap H(y, V) = \emptyset$ unless $x \in V$ or $y \in U$. We can assume that $H(x, U) \subset H(x, W)$ if $U \subset W$.

Metric spaces are monotonically normal and monotonically normal spaces are countably paracompact [8] and collectionwise normal. But there are no implications between paracompactness and monotone normality. Compact spaces need not be monotonically normal. On the other hand all linearly ordered spaces are monotonically normal and linearly ordered spaces fail to be paracompact if and only if they have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal. (We always assume the order topology on such a cardinal or ordinal.)

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By Theorem I monotonically normal spaces share this property with linearly ordered spaces.

This paper began as an attempt by the second author to find out to what extent monotonic normality sufficed to prove certain "shrinking theorems", which are true in paracompact spaces. It has evolved into a study of how paracompactness can fail in monotonically normal spaces. She proved what is now:

Corollary 2.1 (d). *Every monotonically normal space in which every increasing open cover has an increasing shrinking is paracompact.*

The first author then used Corollary 2.1(d) as a lemma in the proof of a stronger theorem giving a characterization of hereditary paracompactness in monotonically normal spaces. This proof led to a proof of Theorems I and II and many corollaries listed in Section 2. It is proved there, for instance, in Corollary 2.2, that every open cover of a monotonically normal space can be shrunk.

Theorem I. *A monotonically normal space is paracompact if and only if it does not have a closed subspace homeomorphic to a stationary subset of a regular uncountable cardinal.*

Theorem II. *Every open cover of a monotonically normal space X has a σ-disjoint (partial) refinement $\mathcal{V}$ by open sets such that $X - \bigcup \mathcal{V}$ is the union of a discrete family of closed subspaces each homeomorphic to some stationary subset of a regular uncountable cardinal.* (The cardinals may vary with the subspaces.)

A space is called *screenable* if every open cover has a σ-disjoint refinement. Normal, countably paracompact, screenable spaces are paracompact [5]; so Theorem I is a trivial consequence of Theorem II.

1. We now prove that every open cover of a monotonically normal space has the quite remarkable kind of refinement described in Theorem II

Lemma 1.1. *If X is homeomorphic to a subspace of an ordinal, then X satisfies the conclusion of Theorem* II.

Proof. If the lemma fails there is a minimal ordinal κ having a subset K and associated open cover $\mathcal{U}$ of K, for which Theorem II fails. There is a closed cofinal subset F of κ which is order isomorphic to the cofinality of κ. We prove the lemma by showing that K (with $\mathcal{U}$) satisfies Theorem II in case F is countable or $F \cap K$ is nonstationary in κ or $F \cap K$ is stationary in κ and F is uncountable.

If F is countable, κ is the union of a discrete family of open and closed proper subintervals of κ. If $F \cap K$ is nonstationary in κ we can assume that F was chosen so that $F \cap K = \emptyset$; in this case K is contained in the union of a discrete in K family of open in κ proper subintervals of κ. In both cases, the intersection of K with each of the subintervals satisfies Theorem II, so the union which is K also satisfies Theorem II.

So assume that F is uncountable and $F \cap K$ is stationary in κ. For all $\alpha \in F \cap K$ there is $f(\alpha) < \alpha$ with $[f(\alpha), \alpha] \cap K$ contained in a member of $\mathcal{U}$. By the pressing-down lemma, there is $\beta < \kappa$ such that $\gamma < \kappa$ implies $\gamma < \alpha < \kappa$ for some $\alpha \in F \cap K$ with $F(\alpha) = \beta$. Let $\mathcal{A}$ be the set of all maximal intervals of κ in $(\kappa - \beta) - F$. For each $A \in \mathcal{A}$, $A \cap K$ is contained in a single member of $\mathcal{U}$. Thus $\mathcal{V} = \{A \cap K \mid A \in \mathcal{A}\}$ is a partial refinement of $\mathcal{U}$ by disjoint open sets and $(K - \beta) = \bigcup \mathcal{V} = F \cap K$ which is homeomorphic to a stationary subset of a regular uncountable cardinal. Since K is the union of $K - \beta$ and $K \cap \beta$ which are disjoint, open and closed subsets of K both satisfying Theorem II, K does also. $\square$

We next give some technical definitions from which we can state Theorem II′ which we use to prove Theorem II.

For a family $\mathcal{P}$ of subsets of a space X we define a σ-ideal: $\mathcal{I}_{\mathcal{P}} = \{Y \subset X \mid Y$ is the union of a σ-relatively discrete partial refinement of $\mathcal{P}\}$.

A set $\mathcal{X}$ of subsets of X is called *relatively discrete* if it is discrete in the subspace $\bigcup \mathcal{X}$. A relatively discrete family in a monotonically normal space can be extended to a disjoint open family since the space is hereditarily collectionwise normal.

We call a partition $\mathcal{P}$ of X *scattered* if the members of $\mathcal{P}$ are disjoint and there are an ordinal κ and a one-to-one listing $\{P_\alpha \mid \alpha < \kappa\}$ of $\mathcal{P}$ with $\bigcup_{\alpha \leq \beta} P_\alpha$ open for every $\beta < \kappa$.

If $Y \subset X$ for some space X, we say Y is a *GO-ordinal* (subspace of X) provided there is an open neighborhood N of Y and homeomorphism h from $\bar{Y} \cap N$ onto a subspace of an ordinal; we say that such an h *determines* that Y is a GO-ordinal. Observe that if $Z \subset Y$, then $h \restriction \bar{Z} \cap N$ determines that Z is also a GO-ordinal.

Suppose u is a *neighborhood assignment* for X, i.e., a function $u : X \to \{$open sets of $X\}$ with $x \in u(x)$ for all $x \in X$. We say that $Y \subset X$ is u-small if there is a dense subset R of Y and an h which determines that Y is a GO-ordinal such that $h(Y)$ has no last term, and, for all $x \in R$ and $y \in Y$ with $h(y) < h(x)$, $y \in u(x)$. We say that such a pair $\langle R, h\rangle$ *determines* that Y is u-small. Define

$$\mathcal{J}_u = \{J \subset X \mid J \text{ is the union of a } \sigma\text{-relatively discrete family of } u\text{-small subsets of } X\}.$$

Recalling the definition of $\mathcal{I}_{\mathcal{P}}$, define for a cover $\mathcal{P}$ of X, $\mathcal{J}_{u\mathcal{P}} = \{I \cup J \mid I \in \mathcal{I}_{\mathcal{P}}$ and $J \in \mathcal{J}_u\}$.

A subset of a member of $\mathcal{J}_{u\mathcal{P}}$ is in $\mathcal{J}_{u\mathcal{P}}$ since:

Lemma 1.2. *If Y is a GO-ordinal subspace of X, then $Y \in \mathcal{J}_{u\mathcal{P}}$.*

Proof. Assuming the lemma is false, let λ be the minimal ordinal for which there is a GO-ordinal $Y \notin \mathcal{J}_{u\mathcal{P}}$ determined by some h with $h(Y) \subset \lambda$. We can assume that λ has uncountable cofinality. We get a contradiction from both Cases 1 and 2.

Case 1: *$h(Y)$ is not stationary in λ.*

Let L be a closed unbounded subset of λ missing $h(Y)$. Oberve that the set $\mathcal{M}$ of all maximal complementary intervals in $\lambda - L$ is relatively discrete in λ. Also, if $M \in \mathcal{M}$, $h^{-1}(M) \cap Y \in \mathcal{J}_{u\mathcal{P}}$ by the minimality of λ. Thus $Y = \bigcup_{M \in \mathcal{M}} (Y \cap h^{-1}(M)) \in \mathcal{J}_{u\mathcal{P}}$.

Case 2: *$h(Y)$ is stationary in λ.*

By the "pressing-down lemma" there are $y \in Y$ and $R \subset Y$ such that $h(R)$ is cofinal with $h(Y)$ (and λ), and, for all $x \in R$, the intersection of $h(Y)$ with the interval $[h(y), h(x)]$ of λ is contained in $h(u(x) \cap Y)$. We assume that $h(R) \subset \lambda - h(y)$. If $S = h^{-1}(\overline{h(R)})$, clearly $S \cap Y$ is u-small; so $S \cap Y \in \mathcal{J}_{u\mathcal{P}}$. But the same argument given in Case 1, with $\overline{h(S)} \cap \lambda$ instead of L, shows that $Y - S \in \mathcal{J}_{u\mathcal{P}}$. Thus $Y \in \mathcal{J}_{u\mathcal{P}}$. □

Theorem II′. *If X is a monotonically normal space, $\mathcal{P}$ a scattered partition of X, and u a neighborhood assignement for X, then $X \in \mathcal{J}_{u\mathcal{P}}$.*

Proof. Otherwise let κ be the minimal ordinal for which there is a monotonically normal space X, a scattered partition $\mathcal{P} = \{P_\alpha \mid \alpha < \kappa\}$ of X, and a neighborhood assignment u for X such that $X \notin \mathcal{J}_{u\mathcal{P}}$. Fix X, $\mathcal{P}$, and u.

Let H be the "monotone normality operator" for X as described in the first sentence of the paper.

For $\alpha < \kappa$ let $U_\alpha = \bigcup_{\beta \leq \alpha} P_\beta$ and $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$. We say $Y \subset X$ is *bounded* if $Y \subset U_\alpha$ for some $\alpha < \kappa$ (and unbounded otherwise). Observe that κ is a limit ordinal; otherwise $\kappa = \alpha + 1$ and $\bigcup_{\beta < \alpha} P_\beta \in \mathcal{J}_{u\mathcal{P}}$ by the minimality of κ; and $P_\alpha \in \mathcal{J}_{u\mathcal{P}}$ so $X = \bigcup_{\beta < \kappa} P_\beta \in \mathcal{J}_{u\mathcal{P}}$ contradicting $X \notin \mathcal{J}_{u\mathcal{P}}$.

Lemma 1.3. (a) *If $Y \subset X$ is bounded, then $Y \in \mathcal{J}_{u\mathcal{P}}$.*

(b) $\mathcal{J}_{u\mathcal{P}} = \mathcal{J}_{u\mathcal{U}}$.

(c) *κ is a regular uncountable cardinal.*

Proof. (a) follows from the minimality of κ.

(b) $\mathcal{J}_{u\mathcal{P}} \subset \mathcal{J}_{u\mathcal{U}}$ by definition and $\mathcal{J}_{u\mathcal{U}} \subset \mathcal{J}_{u\mathcal{P}}$ by (a).

(c) Since $X \notin \mathcal{J}_{u\mathcal{P}}$, κ is uncountable. Suppose $\gamma = (\text{cofinality of } \kappa) < \kappa$ and let $\{\gamma_\beta \mid \beta < \gamma\}$ be an increasing cofinal sequence in κ. By the minimality of κ, $\bigcup \{U_{\gamma_\beta} \mid \beta < \gamma\} \in \mathcal{J}_{u\mathcal{U}}$. So $X \in \mathcal{J}_{u\mathcal{U}}$; which, by (b), contradicts $X \notin \mathcal{J}_{u\mathcal{P}}$. Thus κ is proved to be regular. □

Let $\mathcal{J} = \mathcal{J}_{u\mathcal{P}} = \mathcal{J}_{u\mathcal{U}}$. From the well-known fact that every point finite open family has a σ-discrete refinement, we get:

Lemma 1.4. *If $\mathcal{Y} \subset \mathcal{J}$ is a point finite family of open sets, then $\bigcup \mathcal{Y} \in \mathcal{J}$.*

For $x \in X$ define $\rho(x)$ to be the unique α with $x \in P_\alpha$.

We say that a family $\mathcal{X}$ of subsets of X is *relatively closed closure preserving* provided, for every $\mathcal{X}' \subset \mathcal{X}$, $\bigcup \mathcal{X}'$ is a closed set in the subspace $\bigcup \mathcal{X}$ of X.

Lemma 1.5. *If $\mathscr{Z}$ is a σ-relatively closed closure preserving family of bounded subsets of X, then $\bigcup \mathscr{Z} \in \mathscr{J}$.*

Proof. Since $\mathscr{J}$ is closed under countable unions, we can assume that $\mathscr{Z}$ is a relatively closed closure preserving family.

Index $\mathscr{Z} = \{Z_\delta \mid \delta < \lambda\}$ and let $Z'_\delta = Z_\delta - \bigcup_{\gamma<\delta} Z_\gamma$. Then let v be the neighborhood assignment for X defined by

$$v(x) = \begin{cases} H\left(x,\ U_{p(x)} \cap u(x) - \overline{\bigcup_{\gamma<\delta} Z_\gamma}\right), & \text{if } x \in Z'_\delta, \\ u(x), & \text{if } x \notin \bigcup \mathscr{Z}. \end{cases}$$

Note that for $\delta < \lambda$, Z_δ is bounded. So, by the minimality of κ, $Z'_\delta \in \mathscr{J}_{v\mathscr{P}}$. Thus $Z'_\delta = Z_{\delta 1} \cup Z_{\delta 2}$ for some $Z_{\delta 1} \in \mathscr{J}_{\mathscr{P}}$ and $Z_{\delta 2} \in \mathscr{J}_v$. Hence $\bigcup \mathscr{Z} \in \mathscr{J}$ provided both:

Claim 1. $\bigcup_{\delta<\lambda} Z_{\delta 1} \in \mathscr{J}$.

Claim 2. $\bigcup_{\delta<\lambda} Z_{\delta 2} \in \mathscr{J}$.

Proof of Claim 1. For each $\alpha < \kappa$ and $n \in \omega$, choose $Z_{\delta\alpha n} \subset P_\alpha$ such that $\{Z_{\delta\alpha n} \mid \alpha < \kappa\}$ is relatively discrete for a fixed n, and $Z_{\delta 1} = \bigcup \{Z_{\delta\alpha n} \mid \alpha < \kappa,\ n \in \omega\}$. For each $\delta < \lambda$ and $n \in \omega$ choose a disjoint open expansion $\{G_{\delta\alpha n} \mid \alpha < \kappa\}$ of $\{Z_{\delta\alpha n} \mid \alpha < \kappa\}$ with $G_{\delta\alpha n} \subset U_\alpha - \overline{\bigcup_{\gamma<\delta} Z_\gamma}$. Using the monotone normality operator H, set $Y_{\delta\alpha n} = \bigcup \{H(x, G_{\delta\alpha n}) \mid x \in Z_{\delta\alpha n}\}$. Let $Y_{\alpha n} = \bigcup \{Y_{\delta\alpha n} \mid \delta < \lambda\}$.

By Lemma 1.4, we need only prove that $\{Y_{\alpha n} \mid \alpha < \kappa\}$ is point finite for every n. Suppose not. Then there is an $x \in X$, $n \in \omega$, and $\alpha_0 < \alpha_1 < \cdots < \kappa$ such that $x \in Y_{\alpha_i n}$ for all $i \in \omega$. For each $i \in \omega$, choose $\delta_i < \lambda$ such that $x \in Y_{\delta_i \alpha_i n}$. Since $\delta_i = \delta_j$ and $\alpha_i \neq \alpha_j$ imply $Y_{\delta_i\alpha_i n} \cap Y_{\delta_j \alpha_j n} = \emptyset$, we can assume that $\delta_0 < \delta_1 < \cdots$. There are $x_0 \in Z_{\delta_0\alpha_0 n}$ and $x_1 \in Z_{\delta_1\alpha_1 n}$ with $x \in H(x_0, G_{\delta_0\alpha_0 n}) \cap H(x_1, G_{\delta_1\alpha_1 n})$. Hence either $x_0 \in G_{\delta_1\alpha_1 n}$ or $x_1 \in G_{\delta_0\alpha_0 n}$. Since $x_0 \in Z_{\delta_0}$ and $\delta_0 < \delta_1$, $x_0 \notin G_{\delta_1\alpha_1 n}$. Since $x_1 \in Z_{\delta_1\alpha_1 n} \subset P_{\alpha_1}$ and $\alpha_0 < \alpha_1$, $x_1 \notin U_{\alpha_0}$ and thus $x_1 \notin G_{\delta_0\alpha_0 n} \subset U_{\alpha_0}$. This contradiction proves Claim 1.

Proof of Claim 2. Since $Z_{\delta 2} \in \mathscr{J}_v$, $Z_{\delta 2} = \bigcup \{Z_{\delta nl} \mid n \in \omega,\ l \in L_{\delta n}\}$ where each $Z_{\delta nl}$ is some v-small subspace of X and $\{Z_{\delta nl} \mid l \in L_{\delta n}\}$ is a relatively discrete family for each $n \in \omega$, $\delta < \lambda$. (By listing the empty set several times if necessary) we can assume that every indexing set $L_{\delta n}$ is the same set L. Let $\{G_{\delta nl} \mid l \in L\}$ be a disjoint open expansion of $\{Z_{\delta nl} \mid l \in L\}$ and let $V_{\delta nl} = G_{\delta nl} \cap \bigcup \{H(x, v(x)) \mid x \in Z_{\delta nl}\}$. Define $\alpha(\delta, n, l) = \sup\{\alpha < \kappa \mid P_\alpha \cap Z_{\delta nl} \neq \emptyset\}$. For every $\alpha < \kappa$ and $n \in \omega$, define $V_{\alpha n} = \bigcup \{V_{\delta nl} \mid \delta < \lambda,\ l \in L,\ \alpha(\delta, n, l) = \alpha\}$.

By the definitions of v and $V_{\delta nl}$, for every $\alpha < \kappa$ and $n \in \omega$, $V_{\alpha n} \subset U_\alpha$. Hence, by Lemma 1.4 we achieve Claim 2 if we show that $\{V_{\alpha n} \mid \alpha < \kappa\}$ is point finite for each n.

So fix n and suppose there is an $x \in V_{\alpha_i n}$ for infinitely many distinct $\alpha_i < \kappa$. We can assume that $\alpha_0 < \alpha_1 < \cdots$. For each i choose $\delta_i < \lambda$ and $l_i \in L$ such that $\alpha(\delta_i, n, l_i) = \alpha_i$ and $x \in V_{\delta_i n l_i}$. Since the α_i are distinct, $\{\langle \delta_i, l_i \rangle \mid i \in \omega\}$ are also distinct. If $\delta_i = \delta_j = \delta$ and $l_i \neq l_j$, then $G_{\delta n l_i} \cap G_{\delta n l_j} = \emptyset$ and thus $V_{\delta n l_i} \cap V_{\delta n l_j} = \emptyset$. Hence the δ_i are all distinct and we can assume that $\delta_0 < \delta_1 < \cdots$.

Choose $x_0 \in Z_{\delta_0 n l_0}$ and $x_1 \in Z_{\delta_1 n l_1}$ with $x \in H(x_0, v(x_0)) \cap H(x_1, v(x_1))$. Thus either $x_0 \in v(x_1)$ or $x_1 \in v(x_0)$. Since $x_0 \in Z_{\delta_0}$ and $\delta_0 < \delta_1$, $x_0 \notin v(x_1)$. So $x_1 \in v(x_0)$.

By $\alpha_0 < \alpha_1$ we can pick $y \in Z_{\delta_1 n l_1}$ with $\rho(y) > \rho(x_0)$. Choose an $\langle R, h \rangle$ which determines that $Z_{\delta_1 n l_1}$ is v-small. Then there is an $r \in R$ with $h(r) > h(y)$ and $h(r) > h(x_1)$. Both x_1 and y are in $v(r)$ by the definition of v-small. Since $v(r) \subset U_{\rho(r)}$, $\rho(r) \geq \rho(y)$.

Since $x_1 \in v(r) \cap v(x_0)$, $x_1 \in H(r, \overline{X - \bigcup_{\gamma < \delta_1} Z_\gamma}) \cap H(x_0, U_{\rho(x_0)})$. Since $x_0 \in Z_{\delta_0}$ and $\delta_0 < \delta_1$, $r \in U_{\rho(x_0)}$. But $\rho(x_0) < \rho(y) \leq \rho(r)$ makes this impossible and we have a contradiction which proves Claim 2 and the lemma. □

For each $n \in \omega$ define $V_n(x)$ by induction with $V_0(x) = H(x, U_{\rho(x)})$ and, for $n > 0$, $V_n(x) = H(x, V_{n-1}(x))$. For $n \in \omega$ let

$$T_n(x) = \{t \in X \mid x \in V_n(t)\}.$$

Lemma 1.6. *If $Y \subset X$ and for every $y \in Y$ there is an $n \in \omega$ with $T_n(y) \cap Y$ bounded, then $Y \in \mathcal{J}$.*

Proof. For every $y \in Y$ and $n \in \omega$, let

$$\tau_n(y) = \sup\{\alpha \mid T_n(y) \cap Y \cap P_\alpha \neq \emptyset\}.$$

Then $\kappa \geq \tau_0(y) \geq \tau_1(y) \geq \cdots$ so we can choose $n_y \in \omega$ and $\tau(y) < \kappa$ with $\tau_{n_y}(y) = \tau_i(y) = \tau(y)$ for all $i \geq n_y$.

For every $y \in Y$, since $\tau(y) < \kappa$ which is uncountable and regular, we can choose $y^+ \subset [Y]^{<\kappa}$ with $y^+ \cap T_i(y) \cap Y \cap P_\alpha \neq \emptyset$ for every $\alpha < \kappa$ and $i \geq n_y$ with $T_i(y) \cap Y \cap P_\alpha \neq \emptyset$.

Thus, if we define $\mathscr{Z} = \{Z \subset [Y]^{<\kappa} \mid z \in Z$ imples $z^+ \subset Z\}$, $\mathscr{Z}$ covers Y. For $Z \in \mathscr{Z}$, let $Z^* = \{y \in Y \mid$ there are $z \in Z$ and $k \in \omega$ such that, for infinitely many $i > k$ in ω, $y \in V_{i-k}(r)$ for some $r \in z^+\}$. Observe that $Z \subset Z^*$ and Z^* is bounded.

Next, for all $n \in \omega$, let $Y_n = \{y \in Y \mid n_y = n\}$. Set $Y_n^0 = \{y \in Y_n \mid \tau(y) = \rho(y)\}$ and $Y_n^* = Y_n - Y_n^0$. Note that $\{Y_n^0 \cap P_\alpha \mid \alpha < \kappa\}$ is relatively discrete since $\beta < \alpha$, $y \in Y_n^0 \cap P_\beta$, and $z \in Y_n^0 \cap P_\alpha$ imply $y \notin V_n(z)$. Thus $\bigcup_{n \in \omega} Y_n^0 \in \mathcal{J}_{\mathcal{P}}$ and, by Lemma 1.5, we can prove the lemma by proving:

Claim. *For each $n \in \omega$, $\mathscr{Z}_n^* = \{Z^* \cap Y_n^* \mid Z \in \mathscr{Z}\}$ is relatively closed closure preserving.*

Proof of Claim. Fix $n \in \omega$ and suppose that $\mathscr{Z}' \subset \mathscr{Z}_n^*$ and $p \in Y_n^* \cap (\overline{\bigcup \mathscr{Z}'})$. We need to prove that $p \in \bigcup \mathscr{Z}'$.

Since $p \in Y_n^*$, $n = n_p$, and $\rho(p) < \tau(p)$, there is a $q \in Y$ with $p \in V_n(q)$ and $\rho(p) < \rho(q)$. Since $p \in \overline{\bigcup \mathscr{Z}'}$ and $V_n(p) \cap V_n(q)$ is open, there is $Z \in \mathscr{Z}$ with $(Z^* \cap Y_n^*) \in \mathscr{Z}'$ and $y \in Z^* \cap Y_n^* \cap H(p, V_n(p) \cap V_n(q))$. Let $z \in Z$ and $k \in \omega$ testify to $y \in Z^*$. We show that $p \in Z^*$. Thus $p \in (Z^* \cap Y_n^*) \subset \bigcup \mathscr{Z}'$.

By definition there are infinitely many $i > k$ in ω such that for each i, $y \in V_{i-k}(r_i)$ for some $r_i \in z^+$. If there are infinitely many $i > k+1$ with $p \in V_{i-k-1}(r_i)$, then $p \in Z^*$ by definition. So we can assume there are $r \in z^+ \subset Z$ and $j > 1$ in ω such that $y \in V_j(r)$ but $p \notin V_{j-1}(r)$.

Since $y \in V_j(r) \cap H(p, V_n(p) \cap V_n(q))$, $V_j(r) = H(r, V_{j-1}(r))$, and $p \notin V_{j-1}(r)$, we have $r \in V_n(p) \cap V_n(q)$. Since $r \in V_n(p)$, $\rho(r) \leq \rho(p) < \rho(q)$.

Since $y \in Y_n \cap V_n(q)$, by the definition of Y_n, for all $m > n$, there is $q_m \in Y$ with $\rho(q) \leq \rho(q_m)$ and $y \in V_m(q_m)$. Since $y \in H(r, V_{j-1}(r)) \cap H(q_m, V_{m-1}(q_m))$, and $\rho(r) < \rho(q) \leq \rho(q_m)$ ensures that $q_m \notin V_{j-1}(r)$, $r \in V_{m-1}(q_m)$.

For those m larger than $n_r + 1$ there must be a $t_m \in r^+$ with $\rho(t_m) = \rho(q_m)$ and $r \in V_{m-1}(t_m) = H(t_m, V_{m-2}(t_m))$. Also $r \in V_n(p) \subset H(p, U_{\rho(p)})$. Since $\rho(p) < \rho(t_m) = \rho(q_m)$, $t_m \notin U_{\rho(p)}$. So $p \in V_{m-2}(t_m)$ for infinitely many m. The fact that $r \in z^+ \subset Z$ and thus $t_m \in r^+ \subset Z$ then guarantees that $p \in Z^*$. □

We call a set $Y \subset X$, *n-homogeneous* provided y and z in Y imply $V_n(y) \cap V_n(z) \neq \emptyset$. Observe that, if $n > 0$ and $\rho(y) < \rho(z)$, then $y \in V_{n-1}(z)$ since $V_n(y) = H(y, V_{n-1}(y))$ and $V_n(z) = H(z, V_{n-1}(z))$. Also, clearly, $T_n(x)$ is n-homogeneous for every $x \in X$ and $n \in \omega$.

Fix $x \in X$ and define $\{S_n(x) \mid n \in \omega\}$ by induction, setting $S_0(x) = \{x\}$ and $S_{n+1}(x) = \{y \in X \mid V_0(y) \cap S_n(x) \neq \emptyset\}$. Let $S(x) = \bigcup \{S_n(x) \mid n \in \omega\}$. Note that $V_0(y) \cap S(x) \neq \emptyset$ implies $y \in S(x)$.

Let $\{x_\beta \mid \beta < \lambda\}$ be a one-to-one listing of $\{x \in X \mid T_n(x)$ is unbounded for all $n \in \omega\}$. By induction, for each $\beta < \lambda$, we define $E_\beta \subset X$ by:

$$E_\beta = \bigcup \{Y \subset X \mid Y \text{ is a 2-homogeneous unbounded subset of } S(x_\beta) - \bigcup \{S(x_\gamma) \mid \gamma < \beta \text{ and } E_\gamma \neq \emptyset\}\}$$

if $T_2(x_\beta) \cap S_{x_\gamma} = \emptyset$ for all $\gamma < \beta$ with $E_\gamma \neq \emptyset$; and $E_\beta = \emptyset$ otherwise.

Let $B = \{\beta < \lambda \mid E_\beta \neq \emptyset\}$.

Lemma 1.7. (a) *$\beta \in B$ implies $T_2(x_\beta) \subset E_\beta$.*

(b) *$\beta \in \lambda - B$ implies there is $\alpha < \kappa$ and $\gamma \in B \cap \beta$ such that $(T_2(x_\beta) - U_\alpha) \subset E_\gamma$.*

(c) *For every $x \in X$, $V_0(x)$ intersects at most one of $\{E_\beta \mid \beta \in B\}$.*

(d) *$\{\bar{E}_\beta \mid \beta \in B\}$ is discrete.*

Proof. (a) $T_2(x_\beta)$ is 2-homogeneous, unbounded, and $T_2(x_\beta) \subset T_0(x_\beta) = S_1(x_\beta) \subset S(x_\beta)$. Since $\beta \in B$, $T_2(x_\beta) \cap \bigcup_{\gamma \in \beta \cap B} S(x_\gamma) = \emptyset$. So $T_2(x_\beta) \subset E_\beta$.

(b) Since $\beta \notin B$, there is a minimal $\gamma < \beta$ with $\gamma \in B$ such that $T_2(x_\beta) \cap S(x_\gamma) \neq \emptyset$. Choose $z \in T_2(x_\beta) \cap S(x_\gamma)$ and let $\alpha = \rho(z) + 1$. We show $Y = (T_2(x_\beta) - U_\alpha) \subset E_\gamma$. Y is a 2-homogeneous unbounded set and, by the minimality of γ, $Y \cap \bigcup_{\delta \in \gamma \cap B} S(x_\delta) = \emptyset$. Thus $Y \subset E_\gamma$ provided $Y \subset S(x_\gamma)$.

To see this suppose $y \in Y$. Then $x_\beta \in V_2(y) \cap V_2(z)$ and $\rho(z) < \rho(y)$. Since $y \notin V_1(z)$, $z \in V_1(y)$. Therefore $V_0(y) \cap S(x_\gamma) \neq \emptyset$ and $y \in S(x_\gamma)$.

(c) Suppose $x \in X$. If $x \notin \bigcup_{\beta \in B} S(x_\beta)$, then $V_0(x) \cap \bigcup_{\beta \in B} S(x_\beta) = \emptyset$ and $V_0(x) \cap E_\beta = \emptyset$ for all $\beta \in B$.

So suppose there is a minimal $\beta \in B$ such that $V_0(x) \cap S(x_\beta) \neq \emptyset$. By definition, $V_0(x) \cap E_\gamma = \emptyset$ for all $\gamma < \beta$ in B. We assume $\gamma > \beta$ in B and prove $V_0(x) \cap E_\gamma = \emptyset$.

Suppose on the contrary that $y \in V_0(x) \cap E_\gamma$. Let $Y \subset E_\gamma$ be a 2-homogeneous unbounded set containing y. Choose $z \in Y$ with $\rho(x) < \rho(z)$ and $\rho(y) < \rho(z)$. Then $y \in V_0(x) \cap V_1(z)$, so $x \in V_0(z)$. Since $V_0(z) \cap S(x_\beta) \neq \emptyset$, $z \in S(x_\beta)$. But this contradicts $z \in E_\gamma$ for some $\gamma > \beta$.

(d) follows from (c). □

Lemma 1.8. *Suppose $k \in \omega$. If $\beta < \lambda$, $j \in \omega$, $x \in S_{k+1}(x_\beta)$, $y \in T_{k+j+1}(x_\beta)$, and $\rho(x) < \rho(y)$, then $x \in V_j(y)$.*

Proof. Keeping β fixed, we prove the lemma by induction on k:

If $k=0$, take $j \in \omega$, $x \in S_1(x_\beta)$, $y \in T_{j+1}(x_\beta)$, and $\rho(x) < \rho(y)$. Since $x \in S_1(x_\beta) = T_0(x_\beta)$, $x_\beta \in V_0(x) = H(x, U_{\rho(x)})$. Since $y \in T_{j+1}(x_\beta)$, $x_\beta \in V_{j+1}(y) = H(y, V_j(y))$. By $\rho(x) < \rho(y)$, $y \notin U_{\rho(x)}$, so $x \in V_j(y)$.

Assuming the lemma for k, we prove it for $k+1$. Suppose $j \in \omega$, $x \in S_{k+2}(x_\beta)$, $y \in T_{k+j+2}(x_\beta)$ and $\rho(x) < \rho(y)$. Since $x \in S_{k+2}(x_\beta)$, there is $z \in V_0(x) \cap S_{k+1}(x_\beta)$. Applying the lemma for k to z with $j+1$ instead of j, we get $z \in V_{j+1}(y)$. Since $z \in V_0(x) \cap V_{j+1}(y)$, exactly as in case $k=0$, $x \in V_j(y)$. □

Now fix $\beta \in B$. The aim of Lemmas 1.9–1.13 is to establish that $\bar{E}_\beta \in \mathscr{J}$. For every $\gamma < \kappa$, define $D_\gamma = \{x \in \bigcup_{\alpha<\gamma} (E_\beta \cap P_\alpha) \mid \rho(x) \geq \gamma\}$. Let $\Gamma = \{\gamma < \kappa \mid D_\gamma \neq \emptyset\}$.

For $n \in \omega$ and $\delta < \gamma < \kappa$, let $(\delta, \gamma)_n = T_{n+2}(x_\beta) \cap \bigcup_{\delta<\alpha<\gamma} P_\alpha$.

Lemma 1.9. *If $\gamma \in \Gamma$ and $x \in D_\gamma$, then for every open neighborhood U of x, there are $\delta < \gamma$ and $n \in \omega$ with $(\delta, \gamma)_n \subset U$.*

Proof. Since $x \in D_\gamma$, there are $\delta < \gamma$ and $z \in E_\beta \cap P_\delta \cap H(x, U)$. Since $z \in E_\beta \subset S(x_\beta)$, there is an $n \in \omega$ such that $z \in S_{n+1}(x_\beta)$. We show that $(\delta, \gamma)_n \subset U$. To see this, let $y \in (\delta, \gamma)_n$. Thus $y \in T_{n+2}(x_\beta) \cap P_\alpha$ for some α with $\delta < \alpha < \gamma$. By Lemma 1.8, applied to z with $k=n$ and $j=1$, $z \in V_1(y)$. Since $\alpha = \rho(y) < \gamma \leq \rho(x)$ and $z \in V_1(y) \cap H(x, U)$, $y \in U$. □

For $n \in \omega$ define $A_n = \{\alpha < \kappa \mid T_{n+2}(x_\beta) \cap P_\alpha \neq \emptyset\}$. Since $T_{n+2}(x_\beta)$ is unbounded, $|A_n| = \kappa$. Let $A'_n = \{\alpha < \kappa \mid \alpha$ is a limit of A_n in the order topology on $\kappa\}$; then let $A = \bigcap_{n \in \omega} A'_n$. Observe that $\delta < \alpha \in A$ implies $(\delta, \alpha)_n \neq \emptyset$ for all $n \in \omega$, and A is closed and unbounded in κ.

Lemma 1.10. *If $\gamma \in \Gamma \cap A$, then D_γ consists of a single point d_γ.*

Proof. Since $\gamma \in \Gamma$, $D_\gamma \neq \emptyset$. Since $\gamma \in A$, $(\delta, \gamma)_n \neq \emptyset$ for any $n \in \omega$ and $\delta < \gamma$. Since $(\delta, \gamma)_{n+1} \subset (\delta, \gamma)_n$ for all n, if there were two points in D_γ, Lemma 1.9 would contradict the Hausdorffness of X. □

Consider $C_1 = \{\gamma < \kappa \mid \delta \in \Gamma \cap A \cap \gamma$ implies $\rho(d_\delta) < \gamma\}$. Since C_1 is closed and unbounded in κ, $C = C_1 \cap \Gamma \cap A$ is closed in Γ. Note that $d_\delta \neq d_\gamma$ for distinct δ and γ in C.

Lemma 1.11. *$D=\{d_\gamma \mid \gamma \in C\}$ is closed in X (possibly empty).*

Proof. Suppose $x \in \bar{D}-D$. Choose a neighborhood W of x which misses $d_{\rho(x)}$ in case $\rho(x) \in C$. Since $x \in \bar{D}-D$, $U_{\rho(x)} \cap W$ must intersect D and thus we can define $\gamma = \sup(C \cap \rho(x))$. We prove that $x \in D_\gamma$. Then $\gamma \in \Gamma$ and, since C is closed in Γ, $\gamma \in C$. Thus $x = d_\gamma \in D$ contrary to assumption. So it suffices to prove that every neighborhood of x intersects $\bigcup\{E_\beta \cap P_\alpha \mid \alpha < \gamma\}$.

Suppose V is a neighborhood of x contained in $U_{\rho(x)} \cap W$. Since $x \in \bar{D}$ there must be a $\delta \in C$ with $d_\delta \in V$. Since $\delta \in C \cap \rho(x)$, $\delta \leq \gamma$. Since $d_\delta \in V \cap \bigcup\{E_\beta \cap P_\alpha \mid \alpha < \delta\}$, $V \cap E_\beta \cap P_\alpha \neq \emptyset$ for some $\alpha < \delta \leq \gamma$, as desired. □

Remark. One also sees by the above proof that for every $\gamma \in C$, $\{d_\alpha \mid \alpha \in C \cap (\gamma+1)\}$ is closed.

Lemma 1.12. *If $\gamma \in C$, then for every neighborhood U of d_γ, there is a $\delta < \gamma$ with $\{d_\alpha \mid \alpha \in C$ and $\delta < \alpha < \gamma\} \subset U$.*

Proof. By Lemma 1.9, there are $\delta < \gamma$ and $n \in \omega$ such that $(\delta, \gamma)_n \subset H(d_\gamma, U)$. If $\alpha \in C$ has $\delta < \alpha < \gamma$, then again by Lemma 1.9, there are $\delta \leq \delta' < \alpha$ and $n' > n$ in ω such that $(\delta', \alpha)_{n'} \subset V_0(d_\alpha)$. Since $\alpha \in A$, $(\delta', \alpha)_{n'} \neq \emptyset$; so $H(d_\gamma, U) \cap V_0(d_\alpha) \neq \emptyset$. Note that since γ and α are in C, $\rho(d_\alpha) < \gamma \leq \rho(d_\gamma)$; so $d_\gamma \notin U_{\rho(d_\alpha)}$. Thus $d_\alpha \in U$. □

Lemma 1.13. *The set $\bar{E}_\beta$ is an element of $\mathscr{J}$.*

Proof. By Lemmas 1.11 and 1.12, D is a GO-ordinal subspace of X. Thus $D \in \mathscr{J}$ by Lemma 1.2 and hence it suffices to prove that $\bar{E}_\beta - D$ can be covered by a relatively discrete collection of bounded sets. We shall assume that Γ is unbounded in κ. The case when Γ is bounded is similar. For $\gamma \in (\bar{\Gamma} \cap A \cap C_1) \cup \{0\}$, let γ^+ denote the successor of γ in $\bar{\Gamma} \cap A \cap C_1$ and let $F_\gamma = \bigcup\{(\bar{E}_\beta - D) \cap P_\alpha \mid \gamma \leq \alpha < \gamma^+\}$. Let $\mathscr{F} = \{F_\gamma \mid \gamma \in (\bar{\Gamma} \cap A \cap C_1) \cup \{0\}\}$ and observe that $\bigcup \mathscr{F} = \bar{E}_\beta - D$ and $\mathscr{F}$ consists of bounded sets. Thus it is sufficient to prove that $\mathscr{F}$ is relatively discrete. Consider any $x \in \bar{E}_\beta - D$. There is a unique γ such that $x \in F_\gamma$. Observe that if $\rho(x) = \alpha$, then $\gamma \leq \alpha < \gamma^+$. Suppose that $x \in \overline{\bigcup_{\mu<\gamma} (E_\beta \cap P_\mu)}$. Since $\gamma \leq \rho(x)$, $x \in D_\gamma$ and hence $\gamma \in \Gamma$. Thus $\gamma \in C$ which implies that $x = d_\gamma \in D$, a contradiction. Therefore $U_\alpha - (\overline{\bigcup_{\mu<\gamma} (E_\beta \cap P_\mu)}) = V$ is an open neighborhood of x. It is now easy to check that $V \cap F_{\gamma'} = \emptyset$ whenever $\gamma' \neq \gamma$. □

Proof of Theorem II′ (continued). Observe from Lemmas 1.7(d) and 1.13, $E = \bigcup_{\beta \in B} \bar{E}_\beta \in \mathscr{J}$. If $y \in Y = X - E$ and $y \neq x_\beta$ for any $\beta < \lambda$, then $T_n(y)$ is bounded for some n by definition. If $y = x_\beta$ for some $\beta \in \lambda - B$, then by Lemma 1.7(b), $T_2(y) \cap Y$ is bounded. Thus, by Lemma 1.6, $Y \in \mathscr{J}$ and $X = Y \cup E \in \mathscr{J}$. □

Proof of Theorem II. Keep the notation of this section through the statement of Theorem II′, but forget the notation of its proof.

Let X be a monotonically normal space and $\mathcal{U}$ an open cover of X. Without loss of generality we assume that $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$ for some cardinal κ so that $P_\alpha = U_\alpha - \bigcup_{\beta<\alpha} U_\beta \neq \emptyset$ for any $\alpha < \kappa$. We assume also that κ is uncountable since Theorem II is trivial for countable covers. Choose a function $u : X \to \mathcal{U}$ with $x \in u(x)$ for all $x \in X$.

We introduce a final σ-ideal:

$$\mathcal{K} = \{Y \subset X \mid Y \text{ can be covered by a } \sigma\text{-disjoint partial refinement of } \mathcal{U} \text{ by open sets}\}.$$

Since X is hereditarily collectionwise normal, $\mathcal{I}_{\mathcal{P}} \subset \mathcal{K}$; also $u(x) \in \mathcal{K}$ for all $x \in X$.

Lemma 1.14. *If W is open in X, then there is a closed $Z \subset W$ with $W - Z \in \mathcal{K}$.*

Proof. Let v be the neighborhood assignment defined by:

$$v(x) = \begin{cases} H(x, u(x) \cap W), & \text{if } x \in W, \\ H(x, H(x, u(x))), & \text{if } x \notin W. \end{cases}$$

By Theorem II′, $X = X_1 \cup X_2$ where $X_1 \in \mathcal{I}_{\mathcal{P}} \subset \mathcal{K}$ and $X_2 \in \mathcal{I}_v$. Let $\mathcal{F}$ be a σ-relatively discrete family of v-small subspaces of X such that $\bigcup \mathcal{F} = X_2$.

For $F \in \mathcal{F}$, let $F' = F \cap W$. Since $\mathcal{F}$ is a σ-relatively discrete family and X is hereditarily collectionwise normal, $\bigcup\{F' \mid F \in \mathcal{F} \text{ and } F' \in \mathcal{K}\} \in \mathcal{K}$.

For $F \in \mathcal{F}$ there is $\langle R_F, h_F \rangle$ determining that F is v-small. If there is $z \in F'$ such that $\{y \in F' \mid h_F(z) < h_F(y)\} \in \mathcal{K}$, then there is $r \in R_F$ with $h_F(z) < h_F(r)$ and $\{y \in F' \mid h_F(y) \leq h_F(r)\} \subset v(r) \subset u(r) \in \mathcal{K}$; thus $F' \in \mathcal{K}$.

Let $Y = \bigcup\{F' \mid F \in \mathcal{F} \text{ and } F' \notin \mathcal{K}\}$. We show $\bar{Y} \subset W$; so $Z = \bar{Y}$ has the desired property for the lemma. To see that $\bar{Y} \subset W$, we show that $v(x) \cap Y = \emptyset$ for all $x \in X - W$.

Suppose instead that there are $x \in X - W$, $F \in \mathcal{F}$ with $F' \notin \mathcal{K}$, and $z \in v(x) \cap F'$. We reach a contradiction by showing that

$$\{y \in F' \mid h_F(z) < h_F(y)\} \subset u(x) \in \mathcal{K}.$$

Suppose on the contrary that there is $y \in F' - u(x)$ with $h_F(z) < h_F(y)$. Since y belongs to the closure of $\{r \in R_F \mid h_F(z) < h_F(r)\}$, there is an $r \in R_F \cap W - H(x, u(x))$ with $h_F(z) < h_F(r)$. Then $z \in v(r) \cap v(x) = H(r, u(r) \cap W) \cap H(x, H(x, u(x)))$. Since $x \notin W$ and $r \notin H(x, u(x))$ we have a contradiction. □

Lemma 1.15. *If $\mathcal{F}$ is a relatively discrete family of u-small subsets of X, there is $K \in \mathcal{K}$ such that $\{\overline{F - K} \mid F \in \mathcal{F}\}$ is a discrete family of closed GO-ordinal subspaces of X.*

Proof. For each $F \in \mathcal{F}$ there is $\langle R_F, h_F \rangle$ determining that F is u-small and there is an open N_F such that the domain of h_F is $\bar{F} \cap N_F$, a GO-ordinal subspace of N_F. Since $\mathcal{F}$ is relatively discrete and X is hereditarily collectionwise normal, we can assume that $\{N_F \mid F \in \mathcal{F}\}$ are disjoint. Let $W = \bigcup_{F \in \mathcal{F}} N_F$ and let $Z \subset W$ be a closed subset of X, as guaranteed by Lemma 1.14, such that $K = W - Z \in \mathcal{K}$. Note that for every $F \in \mathscr{F}$, $\overline{F-K} \subset \bar{F} \cap Z \subset \bar{F} \cap N_F$. So $\{\overline{F-K} \mid F \in \mathscr{F}\}$ is a closed discrete family of GO-ordinals in X. □

Lemma 1.16. *If $\{F_n \mid n \in \omega\}$ is a family of closed subsets of X, there is a closed discrete family $\{F'_n \mid n \in \omega\}$ such that $\bigcup_{n\in\omega} F_n - \bigcup_{n\in\omega} F'_n \in \mathscr{K}$ and $F'_n \subset F_n$ for all $n \in \omega$.*

Proof. Note that Lemma 1.14 says that for every closed set Z in X, there is an open $N \supset Z$ with $N - Z \in \mathscr{K}$. So for every $n \in \omega$ choose an open $W_n \supset F_n$ such that $W_n - F_n \in \mathscr{K}$. Let $W = \bigcup_{n\in\omega} W_n$ and let F be a closed subset of W with $W - F \in \mathscr{K}$.

For $n \in \omega$, let $F'_n = F \cap F_n - \bigcup_{i<n} W_i$.

Observe that $\bigcup_{n\in\omega} F_n - \bigcup_{n\in\omega} F'_n \in \mathscr{K}$ since for every n,

$$F_n - \bigcup_{m\in\omega} F'_m = (F_n - F) \cup \left(F \cap F_n - \bigcup_{m\in\omega} F'_m\right)$$
$$\subset (W - F) \cup \bigcup_{i<n} (W_i - F_i) \in \mathscr{K}.$$

By definition $\mathscr{F}' = \{F'_n \mid n \in \omega\}$ is a family of disjoint closed subsets of X. Let us show that $\mathscr{F}'$ is locally finite which will complete the proof of the lemma.

Suppose $x \in X$. If $x \notin W$, $X - F$ is a neighborhood of x missing $\bigcup \mathscr{F}'$. So suppose $x \in W_n$; then W_n is a neighborhood of x missing all F'_m for $m > n$. □

Proof of Theorem II (continued). It suffices to find a $K \in \mathscr{K}$ such that $X - K$ is the union of a discrete family of closed GO-ordinal subspaces of X: then an application of Lemma 1.1 yields the theorem.

Since Theorem II′ holds and $\mathscr{I}_{\mathscr{P}} \subset \mathscr{K}$, $X = X_1 \cup X_2$ where $X_1 \in \mathscr{K}$ and X_2 is the union of a σ-relatively discrete family of u-small subspaces of X. By Lemma 1.15, for each $n \in \omega$, there is $K_n \in \mathscr{K}$ and a discrete family $\mathscr{F}_n$ of closed GO-ordinal subspaces of X such that $X_2 \subset \bigcup_{n\in\omega} (K_n \cup \bigcup \mathscr{F}_n)$. Observe that $\bigcup_{n\in\omega} K_n \in \mathscr{K}$.

Defining $F_n = \bigcup \mathscr{F}_n$ and applying Lemma 1.16 to $\{F_n \mid n \in \omega\}$ yields a discrete family $\{F'_n \mid n \in \omega\}$ of closed sets with $F'_n \subset F_n$ and $\bigcup_{n\in\omega} F_n - \bigcup_{n\in\omega} F'_n \in \mathscr{K}$. Thus, if we define $\mathscr{F}' = \bigcup_{n\in\omega} \{F'_n \cap F \mid F \in \mathscr{F}_n\}$, we have a discrete family of closed GO-ordinal subspaces of X with $X - \bigcup \mathscr{F}' \in \mathscr{K}$, as desired. □

2. Some corollaries

From Theorem I it follows that any condition which makes "GO-spaces" paracompact also makes monotonically normal spaces paracompact. A list of sixteen such conditions appears in [4].

In Corollary 2.1 we list some of the paracompactness theorems for monotonically normal spaces which answer some questions posed in the literature:

Corollary 2.1. *A monotonically normal space is paracompact if one of the following conditions hold*:

(a) *X is weakly $\delta\theta$-refinable (i.e., weakly submeta-Lindelöf).*

(b) *X is perfectly normal.*
(c) *X has a point countable separating open cover.*
(d) *Every increasing open cover of X has an increasing shrinking.*

Proof. A stationary subspace of a regular uncountable cardinal has none of these properties, so Theorem I immediately yields the corollary. □

Remark. Corollary 2.1(a)-(c) answer, in order, Questions 7.1-7.3 from [7]. Corollary 2.1(d) answers a question from [3]; note that property (d) does *not* imply paracompactness in just normal spaces [1, 6].

Corollary 2.2 follows from Theorem II. We give a proof although it is not difficult. (See [8] for a discussion of shrinking properties.)

Corollary 2.2. *Every open cover of a monotonically normal space can be shrunk.*

Proof. Let X be our space and $\mathcal{U}=\{U_\alpha \mid \alpha<\kappa\}$ be our open cover. It suffices to find, for every $\alpha<\kappa$, closed sets Y_α and Z_α contained in U_α such that $\{Y_\alpha \cup Z_\alpha \mid \alpha<\kappa\}$ covers X.

By Theorem II there is a set $\mathcal{V}=\bigcup_{n\in\omega}\mathcal{V}_n$ where each $\mathcal{V}_n$ is a partial refinement of $\mathcal{U}$ by disjoint open sets such that $X-\bigcup\mathcal{V}=\bigcup\mathcal{F}$ for some discrete family $\mathcal{F}$ of closed sets each homeomorphic to a stationary subset of a regular uncountable cardinal. Without loss of generality, for each $n\in\omega$, we can index $\mathcal{V}_n=\{V_{\alpha n}\mid \alpha<\kappa\}$ with $V_{\alpha n}\subset U_\alpha$. Let $V_n=\bigcup\mathcal{V}_n$.

Since X is collectionwise normal we can choose an open neighborhood W_F of F for each $F\in\mathcal{F}$ with $\{\bar{W}_F \mid F\in\mathcal{F}\}$ discrete. Every open cover of an $F\in\mathcal{F}$ can be shrunk; so $F=\bigcup_{\alpha<\kappa}F_\alpha$ where F_α is a closed subset of U_α. Choose an open W_{F_α} with $F_\alpha\subset W_{F_\alpha}\subset \bar{W}_{F_\alpha}\subset U_\alpha\cap W_F$. Let $W_\alpha=\bigcup_{F\in\mathcal{F}}W_{F_\alpha}$ and $W=\bigcup_{\alpha<\kappa}W_\alpha$.

Since $\{V_n \mid n\in\omega\}\cup W$ is countable open cover of the normal and countably paracompact space X, we can find a locally finite closed cover $\{K_n \mid n\in\omega\}\cup J$ of X with all $K_n\subset V_n$ and $J\subset W$.

Then $Y_\alpha=\bigcup_{n\in\omega}(V_{\alpha n}\cap K_n)$ and $Z_\alpha=\bar{W}_\alpha\cap J$ are closed subsets of U_α and $\bigcup_{\alpha<\kappa}(Y_\alpha\cup Z_\alpha)=X$, as desired. □

In fact Theorems I and II allow one to obtain a number of results for various subclasses of monotonically normal spaces. We give a (very incomplete) sample of such results below.

Corollary 2.3. (a) *A Čech-complete monotonically normal space is paracompact if and only if it does not have a closed subspace homeomorphic to a regular uncountable cardinal.*

(b) *A Čech-complete monotonically normal space is hereditarily paracompact if and only if it does not contain a subspace homeomorphic to* ω_1.

(c) [9] *A compact monotonically normal space is hereditarily paracompact if and only if it has countable tightness.*

(d) *A locally hereditarily Lindelöf monotonically normal space has an open paracompact subspace whose complement is the union of a discrete family of closed subspaces each homeomorphic to some stationary subset of* ω_1.

(e) *A manifold of dimension* $\geqslant 2$ *is metrizable if and only if it is monotonically normal.*

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